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# BIOGRAPHY.

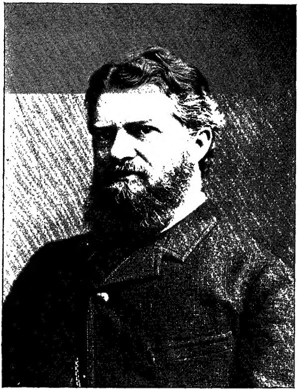
## SIMON NEWCOMB, PH. D., LL. D.

BY J. M. COLAW.

SIMON NEWCOMB was born in Wallace, Nova Scotia, in 1835. After being educated by his father he engaged for some time in teaching. He came to the United States in 1853, and was engaged for two years as a teacher in Maryland. There he became acquainted with Joseph Henry and Julius E. Hilgard, who recognizing his aptitude for mathematics, secured his appointment in 1857 as computer on the "Nautical Almanac," which was then published in Cambridge, Mass. In Cambridge he came under the influence of Professor Benjamin Peirce. He entered the Lawrence Scientific School and was graduated in 1858, continuing thereafter for three years as a graduate student.

In 1861 he was appointed professor of mathematics in the U. S. Navy and assigned to duty at the U. S. Naval Observatory in Washington. There he negotiated the contract for the 26-inch equatorial telescope authorized by congress, supervised its construction and planned the tower and dome in which it is mounted.

He was chief director of the commission created by congress to observe the transit of Venus on December 8, 1874. He visited the Saskatchewan region in 1860 to observe an eclipse of the Sun, and in 1870-1 was sent to Gibralter for a similar purpose. In 1882 he commanded an expedition to observe the transit of Venus at the Cape of Good Hope. Meanwhile in 1887 he became senior professor of mathematics in the U. S. Navy, and since that time has been in charge of the office of the "American Ephemeris and Nautical Almanac". Professor Newcomb has a large corps of assistants in Washington.



Simon Newcomb, Ph. D., LL. D.

In addition to these duties, in 1884 he became professor of mathematics and astronomy in Johns Hopkins, (succeeding the distinguished Sylvester, upon the departure of the latter to accept a professorship at Oxford), where he has had charge of the *American Journal of Mathematics.* However he is not now editor of that Journal, having recently severed his immediate active connection with the Johns Hopkins University for the next two or three years.

Professor Newcomb has been intimately associated with the equipment of the Lick observatory of California, and examined the glass of the great telescope and its mounting before its acceptance by the trustees.

The results of his scientific work have been given to the world in more than one hundred papers and memoirs. Concerning these, Arthur Caley, president of the Royal Astronomical Society of Great Britain, said: "Professor Newcomb's writings exhibit, all of them, a combination on the one hand of mathematical skill and power and on the other of good hard work, devoted to the furtherance of astronomical science."

His work has been principally in the mathematical astronomy of the solar system, particularly Neptune, Uranus, and the Moon, but the whole plan includes the most exact possible tables of the motions of all the planets. Amongst the most important of his papers are: "On the Secular Variations and Mutual Relations of the Orbits of the Asteroids" (1860); "An Investigation of the Orbit of Neptune, with general tables of its motion" (1874); "Researches on the Motion of the Moon" (1876); "Measure of the Velocity of Light" (1884); and "Development of the Purturbative Function and its Derivative in the Sines and Cosines of the Eccentric Anomaly, and in Powers of the Eccentricities and Inclinations" (1884).

In 1874 Columbian University of Washington conferred on him the degree of LL. D., and in 1875 he received the same degree from Yale, also from Harvard in 1874, and from Columbia College in 1887, while on the 300th anniversary of the founding of the University of Leyden in 1875, that institution gave him the degree of Master of Mathematics and Doctor of Natural Philosophy, and on the 500th anniversary of the University of Heidelberg in 1886 he received the degree of Ph. D. Besides the degrees just mentioned he received one from Edinburgh in 1891, one on the occasion of the tercentenary of the University of Dublin in 1892, and one from Paris on the tercentenary of Galileo's connection with the University in 1893.

He was awarded the gold medal of the Royal Astronomical Society in 1874 and in 1878 received the great gold Huyghens medal of the University of Leyden, which is given to astronomers once in 20 years for the most important work acomplished in that science between its awards. Besides the two gold medals mentioned Professor Newcomb received a third in 1890, the Copley medal, given by the Royal Society of England.

In 1887 the Russian Government ordered the portrait of Professor Newcomb to be painted for the collection of famous astronomers at the Russian observatory at Pulkowa, and also ordered to be presented to him a vase of jasper with marble pedestal seven feet high. The University of Tokyo has also

presented him with two vases of bronze.

He was elected an associate member of the Royal Astronomical Society in 1872, corresponding member of the Institute of France in 1874, and foreign member of the Royal Society 1877; and he also holds honorary or corresponding relations to nearly all the European academies of Science. In 1877 he was elected one of the eight members of the council of the Astronomische Gesellschaft, an international astronomical society that meets once in two years. He was elected to the National Academy of Sciences in 1869 and since 1883 has been its vice president. In 1876 he was elected president of the American Association for the Advancement of Science, and delivered his retiring address at the St. Louis meeting in 1878. He also held the presidency of the American Society for Physical Research.

He was elected member of the New York Mathematical Society in 1891, and delivered an address, entitled "Modern Mathematical Thought" before the annual meeting of the Society, December 28, 1893, which was published in the *Bulletin* of the Society for January 1894, and in *Nature* of February 1, 1894.

Professor Newcomb's book on Popular Astronomy (1877) has been republished in England and translated into German, while "School Astronomy" by Newcomb and Holden (1879), and their "Briefer Course" (1883), are used as text books in most of our colleges.

Professor Newcomb has also carried on important investigations on subjects purely mathematical. An important contribution by him on "Elementary Theorems Relating to the Geometry of a Space of Three Dimensions and of Uniform Positive Curvature in the Fourth Dimension", was published in Borchardt's Journal, Berlin, 1877. Full extracts of this important cotribution to non-Euclidean geometry are given in the Encyclopaedia Britannica, article "Measurment". In Vol. I. of the American Journal of Mathematics he has a note "On a Class of Transformations which Surfaces may Undergo in Space of more than Three Dimensions," in which he shows, for instance, that if a fourth dimension were added to space, a closed material surface (or shell) could be turned in side out by simple flexure without either stretching or tearing. Later articles have been on the theory of errors in observations. In former years he also contributed to the Mathematical Monthly and the Analyst.

He has also written a series of mathematical text-books, comprising Algebra (1881); Geometry (1881); Trigonometry and Logarithms (1882); School Algebra (1882); Analytic Geometry (1884); Essentials of Trigonometry (1884); and Calculus (1887). These works have been favorably received and are everywhere regarded as text-books of decided merits.

Professor Newcomb refers to astronomy as his profession and to political economy as his recreation, and in the latter branch has written several books and a number of magazine articles.

[In the main the foregoing sketch is taken from Appleton's Cyclopaedia of American Biography, Vol. IV., New York, 1888, but some statements

have been incorporated from "The Teaching and History of Mathematics in the U. S., by Professor Florian Cajori (1890), to which has been added such matter as was necessary to bring the sketch down to date.]

## SOME GENERAL FORMULAS FOR SQUARE NUMBERS WITH APPLICATIONS.

By Professor P. H. PHILBRICK, M. S., C E., Lake Charles. Louisiana

Let us first examine the Formula, $(m^2+n^2)^2=(m^2-n^2)^2+(2mn)^2$.....(1).

This is identically true for all values of $m$ and $n$ and expresses one square number as the sum of two other square numbers. Give $m$ and $n$ different values and find as follows:

| $m$ | $n$ | | $m$ | $n$ | |
|---|---|---|---|---|---|
| 2 | 1 | $5^2=3^2+4^2$ | 5 | 1 | $26^2=24^2+10^2$ |
| 2 | 1 | $10^2=8^2+6^2$ | 5 | 2 | $29^2=21^2+20^2$ |
| 3 | 2 | $13^2=5^2+12^2$ | 5 | 3 | $34^2=16^2+30^2$ |
| 4 | 1 | $17^2=15^2+8^2$ | 5 | 4 | $41^2=9^2+40^2$ |
| 4 | 2 | $20^2=12^2+16^2$ | | | |
| 4 | 3 | $25^2=7^2+24^2$ | | | |

In reference to this formula, we remark that in order to produce all possible sets of numbers answering to the conditions, the numbers of no set being equimultiples of those of any other set, then evidently $m$ and $n$ must be prime numbers and cannot both be even.

Moreover $m$ and $n$ cannot both be odd, for in that case $m^2+n^2$, $m^2-n^2$, and $2mn$ would all be even, and of the form $2(m^2_1+n^2_1)$, $2(m^2_1-n^2_1)$ and $2(2m_1n_1)$, and would give numbers twice as large as the values $m$ and $n$ would give. This is shown above; for $m=3$, $n=1$; $m=5$, $n=1$; etc.

Hence, $m$ and $n$ are both prime, one is odd the other even.

Hence, $m^2+n^2$ and $m^2-n^2$ are both odd and $2mn$ is even.

For all sets of values, we have the following *General* formula:

$[p(m^2+n^2)]^2=[p(m^2-n^2)]^2+(2pmn)^2$....(2), in which $m$ and $n$ are prime numbers, one of which is odd and the other even, and $p$ any number. Let $m=2$ and $n=1$; then

$p=2$ gives $10^2=6^2+8^2$ $\quad$ $p=3$ gives $15^2=9^2+12^2$

$p=4$ gives $20^2=12^2+16^2$ $\quad$ $p=5$ gives $25^2=15^2+20^2$, etc.

None of these sets could be given by (1) $m$ and $n$ being prime, and one odd, the other even; and all sets given by (2), $p$ being any number not 2 or a square, could not be given by (1) at all.

II. To find three square numbers, whose sum is a square number.

Making $m^2=a^2_1+a^2_2$ and $n^2=a^2_3$ in eq. (1) gives,

$$(a^2_1+a^2_2+a^2_3)^2=(a^2_1+a^2_2-a^2_3)+(2a_1a_3)^2+(2a_2a_3)^2 \ldots\ldots(3).$$

The same may be at once written by analogy. This furnishes a solution to the problem:

To find a parallelopipedon whose diagonal and edges are integral numbers.

If in (3) we make $a_1=a_2$ we have a solution to the problem: To find a parallelopipedon whose base is a square, and diagonal and edges are integral numbers.

Ex. Let $a_1=a_2=2$ and $a_3=1$. Then $9^2=7^2+4^2+4^2$.

The sides of the base are 4, altitude 7, and diagonal 9. If in (3) we assign any values to $a_1$ and $a_2$, we have a solution to the problem: To find a parallelopipedon whose base is any rectangle, and diagonal and edges integral numbers.

Ex. Let $a_1=2$, $a_2=3$, and $a_3=1$. Then $14^2=12^2+4^2+6^2$ or $7^2=6^2+3^2+2^2$. In the latter, the sides of the base are 2 and 3, altitude 6 and diagonal 7.

III. To find $n$ square numbers, whose sum is a square number.

Make $m^2=a^2_1+a^2_2+\ldots\ldots+a^2_{n-1}$, and $n^2=a^2_n$ in eq. (1) and find directly,

$$(a^2_1+a^2_2+\ldots\ldots a^2_n)^2=(a^2_1+a^2_2+\ldots\ldots+a^2_{n-1}-a^2_n)^2$$
$$+(2a_1a_n)^2+(2a_2a_n)^2+\ldots\ldots(2a_{n-1}a_n)^2 \ldots\ldots(4).$$

We may assign values to $a_1, a_2, \ldots\ldots a_{n-1}$ in (4) so as to satisfy any possible conditions. If we make these quantities equal we will have a square number, equal to the sum of $n$ square numbers of which $n-1$ are equal.

If $a_1=1$, $a_2=2\ldots\ldots a_{n-1}=n-1$, and $a_n=1$, we have, $[1^2+2^2+\ldots\ldots+(n-1)^2+1^2]=[1^2+2^2+\ldots\ldots+(n-1)^2-1^2]+[2^2+4^2+\ldots\ldots 2^2(n-1)^2.]$

IV. To find a square number, which added to *any* given number will make a square number.

Let $S$= the given number, $y^2$ the square number, to be added to make a square number.

Put $S+y^2=x^2$. $\therefore (x+y)(x-y)=S=a\times b$.

Put $x+y=a$ and $x-y=b$, then $S=ab$, $x=\dfrac{a+b}{2}$, and $y=\dfrac{a-b}{2}$.

Since $x$ and $y$ are integral $a$ and $b$ must be both odd, or both even.

$$\therefore\ ab+\left(\frac{a-b}{2}\right)^2=\left(\frac{a+b}{2}\right)^2 \text{ or, } ab+\left(\frac{a-b}{2}\right)^2=\left(\frac{a+b}{2}\right)^2;$$

$$\text{or, } ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2.$$

Hence, any number which is the product of two even or of two odd factors can be expressed as the difference between two squares. Since $S$ is entirely unrestricted, this equation is entirely general, and very prolific in results. For example, $S$ may represent any series, complete or incomplete, or

the sums or differences of several series complete or incomplete. We will give several examples, but an indefinite number could be adduced.

*a.* Take $S=1+3+5+\ldots.$ to $n$ terms, $=n^2$.

1. Let $n=4$, then $S=4^2=8\times2=a\times b$. $\frac{a-b}{2}=3$ and $\frac{a+b}{2}=5$ and we have, $1+3+5+7+3^2=5^2$, or $4^2+3^2=5^2$.

2. Let $n=5$, then $S=5^2=25\times1=a\times b$ and we have, $1+3+5+7+9+12^2=13^2$; or $5^2+12^2=13^2$.

3. Let $n=1000$, then $S=(1000)^2=2500\times400$ and we have $(1000_2)+(1050)^2=(1450)^2$.

*b.* Take $S=1^2+2^2+\ \ldots n^2$.

1. Let $n=5$, then $S=55=55\times1$ or $11\times5$.

Let $a=55$ and $b=1$, and we have, $1+2^2+\ldots.+n^2+27^2=28^2$.

Or let $a=11$ and $b=5$, and we have, $1^2+2^n+\ldots.+n^2+3^2=8^2$.

These are the only *two* solutions of this case.

2. Let $S=1^2+2^2+\ldots.+10^2=385\times1=77\times5=55\times7=35\times11$. Taking $a=35$ and $b=11$, we have $1+2+\ldots.+10^2+12^2=23^2$. There are three other solutions.

*c.* Take $S=(1^2+2^2+\ldots.n^2)-(1+2^2+\ldots.m^2)-p^2-q\pm\ldots.$

1. Let $n=7$, $m=2$, $v^2=5^2$, $q=11$.

Then, $S=3^2+4^2+6^2+7^2-11=99=11\times9$. $\therefore\ 3^2+4^2+6^2+7^2-11+1^2=10^2$.

When $S=$any $(n-1)$ square numbers, this formula furnishes another solution to the problem: To find $n$ square numbers whose sum is a square number.

1. Find eleven square numbers whose sum is a square.

Take $S=1^2+2^2+\ldots.10^2=385=35\times11$.

$\therefore\ 1^2+2^2+\ldots.10^2+12^2=23^2$.

Find fourteen square numbers, whose sum is a square number.

Take $S=1^2+2^2+\ldots.13^2=819\times1=273\times3=91\times9=63\times13=39\times21$.

Using 63 and 13, $1^2+2^2+\ldots.13^2+25^2=33^2$.

Using 91 and 9, $1^2+2^2+\ldots.13^2+41^2=50^2$ etc., etc.

In case $S$, though not a prime number, cannot be separated into two factors, both odd or both even, $S$ must be an even number, in which case subtract any odd square number, say $(2n_1-1)^2$, then factor the remainder and proceed as above.

Example. To find two square numbers, one of which being added to, and the other taken from $S=1^2+2^2+\ldots.16^2=1496$ will make a square number.

$$1496=15^2+1271;\ 1271=41\times31=36^2-5^2.$$
$$\therefore\ S-15^2=36^2-5^2,\ \text{or}\ S-15^2+5^2=36^2,$$
$$\text{or}\ 1^2+2^2+\ldots.14^2+16^2+5^2=36^2.$$

COROLLARY.—If the square number $(2n^2_1-1)^2$ taken from $S$ is one of the square numbers in $S$, as in the above example and which may always be the case, the formula still furnishes the means of finding $n$ square numbers whose sum is a square number.

We may write at once, $(n+1)^2-1^2=n(n+2)$
$$(n+2)^2-2^2=n(n+4)$$
$$(n+m)^2-m^2=n(n+2m).$$

Hence we see the form that two factors must have, in order that their product may be equal to the difference between two squares.

We see that the factors are both odd or both even.

## NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins): Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

### CHAPTER THIRD.

*On the Continuity of Space.*

[Continued from the July Number].

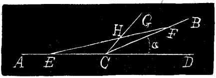
It is here thought best to interpolate some expository matter regarding parts of elementary geometry which involve the difficult idea of *continuity.*

When a mathematician says: "There may be a triangle whose angle-sum differes from a straight angle by less than any *given* finite angle however small," the meaning is simply, "give me geometrically any one particular finite angle you choose, and I will prove geometrically that a triangle may exist the sum of whose three interior angles differs from two right angles by less than that particular angle you have given me."

The problem: "To construct a triangle whose angle-sum differs from a straight angle by less than any given finite angle however small," means, "if any one particular finite angle is given graphically, show how geometrically to construct a triangle whose angle-sum differs from two right angles by less than that one particular *given* finite angle."

The same easy solution of this problem is true both in Euclid and in Lobatschewsky.

Solution: Let $GCD$ be the given finite angle however small. Extend the ray $CD$ through $C$ to $A$. In the ray $CA$ take any point $E$. Join $E$ to any point of the ray $CG$, as $H$. The constructed triangle $ECH$ has an angle sum differing from a straight angle by less than $GCD$.




Proof: The angle $ECH$ differs from a straight angle by the angle $GCD$. Therefore the sum [not greater than a straight angle] of $ECH$ and $CEH$ and $CHE$ differs from a straight angle by less than the given angle $GCD$.

To see that the solution is not restricted by the smallness of the *given* finite angle, notice that if the smaller angle $DCB$ were given we need only join $E$ to any point $F$ in the ray $CB$ to get a triangle $ECF$ which is a solution for the given angle $DCB$.

Thus we have proved that "There may be a triangle whose angle sum differs from a straight angle by less than any given finite angle however small," in the best possible way, namely by showing how to construct it geometrically.

Note that in general a proof that there may be a specified geometric entity, does not necessarily carry with it the possibility of its geometric construction. For example, there may be an angle which is one third of any given finite angle however small, yet is the possibility of its construction so far from being granted, that in fact the trisection of an angle is one of the three great insoluble geometric problems, ranking with the quadrature of the circle and the duplscation of the cube.

But we have just proved that in Euclid as in Lobatschewsky we may actually construct "a triangle whose angle sum is equal to two right angles minus the angle $a$ which is less than any particular given finite angle $b$ however small." In Euclid $a=0$. In Lobatschewsky $a$, though greater than zero is less than the particular given finite angle $b$.

But suppose there were any need for introducing infinitesimal angles, would we be justified in saying "the difference between a finite angle less than two right angles and two right angles is necessarily finite."

Assuredly not. For a straight angle minus an infinitesimal would be finite; but the difference between this finite angle and two right angles would be be only that infinitesinal.

# THE SIMPLEST MODEL FOR ILLUSTRATING THE CONIC SECTIONS.

By LEONARD E. DICKSON, B. Sc., Fellow in Pure Mathematics, in the Chicago University.

The Greek geometers prior to Appollonius of Perga supposed that three cones were necessary in forming the conic sections. Thus, the ellipse was cut from an acute-angled cone; the parabola, from a right-angled cone; the hyperbola, from an obtuse-angled cone, each by a plane perpendicular to an edge. The very names, ellipse, parabola, and hyperbola, express the fact that the angle at the vertex of the cone is less than, equal to, or greater than a right angle; and thus that in the ellipse, the cutting plane *falls short* of the other nappe of the cone; in the parabola, is *parallel* to the edge of the cone; in the hyperbolia, *reaches over* to the other nappe of the cone.

Thus the earlier Greeks thought the ellipse peculiar to the acute-angled cone, the parabola to the right-angled cone, the hyperbola to the obtuse-angled cone.

But Appollonius of Perga showed in the year 250 B. C. that all three conics could be cut from single cone by varying the inclination of the cutting plane. Thus he cut a right-circular cone by a plane. Now if the angle between the cutting plane and the base of the cone be *less* than the angle made by an edge of the cone with the base, the section is an *ellipse*; if *equal*, a *parabola*; if *greater* an *hyperbola*.

*The method of Appollonius, now used in our models consists in cutting a fixed cone by a revolving plane; my method consists in making the plane fixed and revolving the cone.*

Nature furnishes us with a fixed horizontal plane, the surface of a liquid at rest. Filling partly full a hollow glass cone with some liquid (colored to make the effect distinct), we can make at will all the conic sections and their limiting positions. A definite amount of liquid will give a certain series of conics. By varying the quantity of liquid, we can form series of conics magnified in different degrees. Thus we can vary their curvature at will.

Suppose our cone is lying with an edge on a plane. If the plane be horizontal, the section is always a parabola, even if the cone be rolled about on the plane. If the plane is inclined to the horizon at an angle less than the angle at the vertex of the cone, then by rolling the cone about on the plane, the section passes successively through various forms of ellipses, then a parabola, then various forms of hyperbola, and again a parabola, and lastly the original ellipses.

This model has the great superiority over the ordinary wooden models, etc., in that we may readily study by it how the conics pass from one class into another through the limiting forms, the parabola, the circle, and the straight line.

Again, any school boy can make use of his conical ink-bottle, if no better model be at hand.

My method may be hapily extended to the construction of simple models for showing the various sections of any solid whatever.

[Read before the *Texas Academy of Science* in 1892.]

# SERIES OF RATIONAL TRIANGLES.

By SYLVESTER ROBINS, North Branch, New Jersey.

The following is the simplest method known to the writer of finding the dimensions of prime, integral, rational, scalene triangles in series.

PROBLEM: There is an infinite series of rational, scalene triangles in which there is a difference of 1 between the two short sides of every term, and the longest side is one less than double the shortest.

SOLUTION: Let $x$, $x+1$, and $2x-1$ represent the sides.
Then $2x.x(x-1)\ 1=\square$ : $2(x-1)=\square$ : $2x-2=\square$.

$x=\frac{\square}{2}+1=3$ 9 19 33 51 73 99 129 163 201 243 289 339 393 451 &c.
$x+1=4$ 10 20 34 52 74 100 130 164 202 244 290 340 394 452 "
$2x-1=5$ 17 37 65 101 145 197 257 325 401 485 577 677 785 901 "
General expressions: $2n^2+1$, $2(n^2+1)$, $(2n)^2+1$.

PROBLEM: In a certain series of rational △'s there is a constant difference of 2 between two sides, and the third one is 2 less than twice the shortest.

SOLUTION: Let $x$, $x+2$, $2x-2$, represent sides of △: $2x.x.2(x-2)=\square$.
$x=\square+2=3$ 11 27 51 83 123 171 227 291 363 443 531 627 731 &c.
$x+2=5$ 13 29 53 85 125 173 229 293 365 445 533 629 733 "
$2x-2=4$ 20 52 100 164 244 340 452 580 724 884 1060 1252 1460 "
General expressions: $(2n-1)^2+2$, $(2n-1)^2+4$, and $2(2n-1)^2+2$.

PROBLEM: In an infinite series of rational, scalene △'s, there is a difference of 8 in two of the sides, and the other one is 8 less than twice the shortest.

SOLUTION: Represent the sides by $x$, $x+8$, and $2x-8$.
Then $2x.x.8(x-8)=\square$.
$x=\square+8=$ 9 17 33 57 89 129 177 233 297 369 449 537 633 737 &c.
$x+8=17$ 25 41 65 97 137 185 241 305 377 457 545 641 745 "
$2x-8=10$ 26 58 106 170 250 346 458 586 730 890 1066 1258 1466 "
General expressions $(2n-1)^2+2.2^2$, $(2n-1)^2+4^2$, $2\{(2n-1)^2+2^2\}$.

PROBLEM: In an infinite series of rational, scalene △'s, there is a difference of 9 in two of the sides, and the other one is 9 less than twice the shortest.

SOLUTION: $2x.x.9(x-9)=\square$ : $2(x-9)=\square$ : $x-9=\frac{\square}{2}$.

$x=\frac{\square}{2}+9=11$ 17 . 41 59 . 107 137 . 209 251 . 347 401 521 587 &c.
$x+9=20$ 26 . 50 68 . 116 146 . 218 260 . 356 410 530 596 "
$2x-9=13$ 25 . 73 109 . 205 265 . 409 475 . 685 793 1033 1165 "

PROBLEM: In an infinite series of rational, scalene △'s, there is a difference of 18 in two of the sides, and the other one is 18 less than twice the shortest.

SOLUTION: $2x(x-18)\ 18x=\square$: $x=\square+18$.

$x=\square+18=19\ 43\ \ 67\ 139\ 187\ 307\ 379\ \ 547\ \ 643\ \ 859\ \ 979\ 1243$ &c.

$x+18=37\ 61\ \ 85\ 157\ 205\ 325\ 397\ \ 565\ \ 661\ \ 877\ \ 997\ 1261$ "

$2x-18=20\ 68\ 116\ 260\ 356\ 596\ 740\ 1076\ 1268\ 1700\ 1940\ 2468$ "

PROBLEM: In an infinite series of rational, scalene △'s there is a difference of 25 in two of the sides, and the other one is 25 less than twice the shortest.

SOLUTION: $2x.x.25(x-25)=\square$: $2(x-25)=\square$.

$x=\frac{\square}{2}+25=27\ 33\ 43\ 57\ 97\ 123\ 153\ 187\ 267\ 363\ 417$ &c.

$x+25=52\ 58\ 68\ 82\ 122\ 148\ 178\ 212\ 292\ 388\ 442$ "

$2x-25=29\ 41\ 61\ 89\ 169\ 221\ 281\ 349\ 509\ 701\ 809$ "

PROBLEM: In an infinite series of rational, scalene △'s, there is a difference of 32 in two of the sides, and the other one is 32 less than twice the shortest.

SOLUTION: $2x.x.32(x-32)=\square$: $x-32=\square$.

$x=\square+32=33\ 41\ 57\ \ 81\ 113\ 153\ 201\ 257\ 321\ 393$ &c.

$x+32=65\ 73\ 89\ 113\ 145\ 185\ 233\ 289\ 353\ 425$ "

$2x-32=34\ 50\ 82\ 130\ 194\ 274\ 370\ 482\ 610\ 754$ "

PROBLEM: In an infinite series of rational, scalene △'s, there is a difference of 49 in two of the sides, and the other one is 49 less than twice the shortest.

SOLUTION: $2x.x.49(x-49)=\square$: $2(x-49)=\square$.

$x=\frac{\square}{2}+49=\ 51\ \ 57\ \ 67\ \ 81\ \ 99\ 121\ .\ 177\ 211\ 249$ &c.

$x+49=100\ 106\ 116\ 130\ 148\ 170\ .\ 226\ 260\ 298$ "

$2x-49=\ 53\ \ 65\ \ 85\ 113\ 149\ 193\ .\ 305\ 373\ 449$ "

PROBLEM: In an infinite series of rational, scalene △'s, there is a difference of 50 in two of the sides, and the other one is 50 less than twice the shortest.

SOLUTION: $2x.x.50(x-50)=\square$: $x-50=\square$.

$x=\square+50=\ 51\ \ 59\ .\ 99\ \ 131\ 171\ 219\ .\ 339\ 411\ 491\ \ 579$ &c.

$x+50=101\ 109\ .\ 148\ 181\ 221\ 269\ .\ 389\ 461\ 541\ \ 629$ "

$2x-50=\ 52\ \ 68\ .\ 149\ 212\ 292\ 388\ .\ 628\ 772\ 932\ 1108$ "

PROBLEM: In a certain series of rational △'s there is a constant difference of 72 between two sides, and the third one is 72 less than twice the shortest.

SOLUTION: $2x.x.72(x-72)=\square$: $x-72=\square$.

$x=\square+72=\ 73\ \ 97\ 121\ 193\ 241\ 361\ 433\ \ 601\ \ 697\ \ 913$ &c.

$x+72=145\ 169\ 193\ 265\ 313\ 433\ 505\ \ 673\ \ 769\ \ 985$ "

$2x-72=\ 74\ 122\ 170\ 314\ 410\ 650\ 794\ 1130\ 1322\ 1754$ "

# ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

## SOLUTIONS TO PROBLEMS.

22. Proposed by E. S. LOOMIS, A. M., Ph. D., Professor of Mathematics, Baldwin University, Berea, Ohio.

*A* borrows \$1,000 from *B* for 10 years, on which he pays 4% semi-annually.

*A* immediately loans the \$1,000 to *C* for 10 years. who agrees to pay to *A* \$12½ on the first of each month for 120 mos. or 10 yrs., at which time the whole debt is considered canceled, *C* no longer being, in any way, indebted to *A*. Upon the receipt of each of the \$12½ payments made by *C*, *A* immediately reloans it to *D*, E, *F*, etc., upon the same conditions as he loaned the \$1,000 to *C*; at the end of 120 mos. all who are indebted to *A* pay up in full all due him, and he (*A*) pays *B* the principal, all interest having been paid when due.

Query: How many dollars has he in hand?

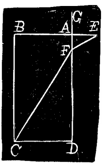
NO SOLUTION RECEIVED.

23. Proposed by H. C. WHITAKER, Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

A rectangular hall 80 feet long, 40 feet wide and 12 feet high, has a spider in one corner of the ceiling. How long will it take the spider to crawl to the opposite corner on the floor, if he crawls a foot in a second on the wall and two feet in a second on the floor?

I. Solution by Professor J. A. TIMMONS, St. Mary's, Kentucky, and the PROPOSER.

It seems to me this problem does not belong to arithmetic. Were the *shortest route* required, arithmetic would solve it; a line from $E$ (the *side* wall supposed to be lying down) to $C$ being the required distance$=95.41$ ft. The time to travel *this* route$=22.02+\frac{73.4}{2}=58.72$ seconds




Were the spider to descend the *end* wall, a line from $G$ to $C$ would give the shortest distance by that route; but although this distance is *longer* than the other one, being 100.32 ft., the *time* would be shorter, being only $13.085+\frac{87.233}{2}=56.691$ seconds. Hence we see that arithmetic alone will not solve it.

Let $F$ be point when spider reaches the floor; call $AF$ $x$; then $FD=80-x$.

We have $EF=\sqrt{x^2+144}$, and $CF=\sqrt{8000-160x+x^2}$; hence the *time* required in seconds$=\sqrt{x^2+144}+\sqrt{2000-40x+\frac{x^2}{4}}$; that is, $u=\sqrt{x^2+144}$

$$+\sqrt{\frac{x^2}{4}-40x+2000}, \text{ is to be a minimum. } \frac{du}{dx}=\frac{x}{\sqrt{(x^2+144)}}+\frac{\frac{x}{4}-20}{\sqrt{\frac{x^2}{4}-(40x+2000)}}.$$

Making this equal to zero and reducing, we get $3x^4-480x^3+25456x^2+23040-921600=0$. By Horner's Method I find $x=5.88$ ft., nearly, $=AF$, $EF=13.36316$=distance down wall, $FC=84.22454$ ft.=distance on floor.

$\therefore$ Time$=13.36316+84.2254\div2=55.47543$ seconds, *Ans.*

II Solution by **F. P. MATZ, M. Sc., Ph. D.**, Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

The spider may take several routes; but the one requiring the *minimum* of time necessitates a perpendicular descent of 12 feet on the wall and a diagonal crossing of $\sqrt{[(80)^2+(40)^2]}=89.44+$feet on the floor, and the time required is 56.72 seconds.

III. Solution by **J. K. ELLWOOD, A. M.**, Principal of Colfax School, Pittsburg, Pennsylvania; **W. L. HARVEY**, Portland, Maine; **P. S. BERG**, Apple Creek, Ohio; **COOPER D. SCHMITT**, Professor of Mathematics, University of Tennessee, Knoxville, Tennessee; and **LINNAEUS HINES**, Teacher in High School, Evansville, Indiana.

The entire distance the spider crawls is the hypotenuse of a rt. triangle whose base is 80 ft. and whose perpendicular is 40+12, or 52 ft., which is $\sqrt{80^2+52^2}$, or 95.41+ft.

The height: the width :: 3 : 10 :: distance crawled on wall : distance on floor. Hence $\frac{3}{13}$ of 95.41+ft., or 22.01 ft. is the distance crawled on wall; and $\frac{10}{13}$ of 95.41+ft., or 73.39ft. is distance crawled on floor.

$\therefore$ $22.01\div1+73.39\div2=58.71+$seconds the time required.

This Problem was also solved by Professor *G. B. M. ZERR*, *FRANK HORN*, *M. A. GRUBER*, and *J. H. DRUMMOND*.

**24.** Proposed by **Mrs. MARY E. HOGSETT**, Danville, Kentucky.

On Jannary 4. 1889, it was noticed that a clock was 15 minutes fast. On March 1, 1894, it was found to be six and one half minutes slow. When and what time was accurate time?

Solution by **FRANK HORN**, Meadville, Missouri.

II. 
1. 1882 days=time from January 4, 1889 to March 1, 1894.
2. 15 minutes+$6\frac{1}{2}$ minutes=time the clock lost in 1882 days.
3. $\frac{1882}{21\frac{1}{2}}$ days=time required for the clock to lose 1 minute.
4. $1313\frac{1}{43}$ days=time required for the clock to lose 15 minutes.
5. January 4, 1889+$1313\frac{1}{43}$ days=33 minutes $29\frac{13}{43}$ seconds past 12 o'clock, A.M., August 11, 1892.

III. $\therefore$ The clock indicated true time, 33 minutes $29\frac{13}{43}$ seconds past 12 o'clock A. M., provided the observation was made at midnight, January 4, 1889.

This problem was also solved by *J. K. ELLWOOD*, *H. C. WHITAKER*, *COOPER D. SCHMITT*, *G. B. M. ZERR*, *F. P. MATZ*, and *P. S. BERG*.

## PROBLEMS.

**29. Proposed by R. H. YOUNG, West Sunbury, Pennsylvania.**

An interest bearing note dated Aug. 1st, 1892, was discounted at 90 days at 8%. The face of the note was \$750, and the proceeds \$759.982. What was the date of discount?

**30. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

While dressing a fifty-cent chicken, a poulterer found a fifty-dollar diamond in the chicken's gizzard. He sold the chicken at a profit of 25cts., changed with good money the counterfeit ten-dollar bill handed him by the unknown purchaser, and realized 50% of the estimated value of the diamond. What per cent. of gain, or loss, did the poulterer make? Suppose the purchaser of the chicken and of the diamond had been one person, what per cent. of gain, or loss, would *he* have made after selling the diamond for \$25 in good money?

**31. Proposed by I L. BEVERAGE, Monterey, Virginia.**

"A man wishes to know how many hogs at \$9, sheep at \$2, lambs at \$1, and calves at \$9 per head, can be bought for \$400, having of the four kinds, 100 animals in all. How many different answers can be given?"

[Satisfactory arithmetical solution desired.]

**32. Proposed by P. C. Cullen, Mead, Nebraska.**

A horse is tied to corner of building 40 feet square, by a rope 110 feet long. Over how much land can he graze?

Solutions to these problems should be received on or before November 1st.

# ALGEBRA.

**Conducted by J. M. Colaw, Monterey, Va. All contributions to this department should be sent to him.**

## SOLUTIONS TO PROBLEMS.

**21. Proposed by Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.**

A tobacconist has two kinds of smoking tobacco, of which the price of the better kind is \$1 and of the inferior \$.75 per ℔. Now, he takes 9 parts of the better and mixes it with two parts of the inferior, then 9 parts of the mixture with two parts of the inferior, etc. What is the price of the $n$th mixture per ℔.?

**I. Solution by H. C. WHITAKER, B. S., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania, and P. H. PHILBRICK, M. S., C. E., Lake Charles, Louisiana.**

The portion of the better in the first mixture was $\frac{9}{11}$, in the second mixture was $(\frac{9}{11})^2$, and in the $n$th mixture was $(\frac{9}{11})^n$, making the portion of

the inferior to be $1-(\frac{9}{11})^n$. The answer is, therefore, $(\frac{9}{11})^n+\frac{3}{4}[1-(\frac{9}{11})^n]=\frac{3}{4}+\frac{1}{4}(\frac{9}{11})^n$ dollars.

**II. Solution by A. L. FOOTE, C. E., Merrick, New York.**

Let $a=(\$1.)$ price of better kind, and $b=(75\text{cts.})$ price of the inferior kind per lb. Then, the value of 1lb. after first mixture is, $a(\frac{9}{11})+b(\frac{2}{11})$; after the second, $[a(\frac{9}{11})+b(\frac{2}{11})]\frac{9}{11}+b(\frac{2}{11})=a(\frac{9}{11})^2+b(\frac{2}{11})(\frac{9}{11}+1)$; after the third, $a(\frac{9}{11})^3+\frac{2b}{11}[(\frac{9}{11})^2+(\frac{9}{11})+1]$, etc., etc.; and after the $n$th mixture, $a(\frac{9}{11})^n+\frac{2b}{11}[(\frac{9}{11})^{n-1}+(\frac{9}{11})^{n-2}+\dots+1]$.

The sum of the series of which $\frac{2b}{11}$ is a factor is $b-b(\frac{9}{11})^n$, and the entire value is $b+(\frac{9}{11})^n(a-b)=\frac{3}{4}+\frac{1}{4}(\frac{9}{11})^n$ dollars.

**III. Solution by the PROPOSER.**

By Finite Differences. Let $f(x)$ be the price per lb. of the $x$th mixture, then $\frac{9}{11}f(x)+\frac{2}{11}\times 75$ will be the price for the $(x+1)^{th}$. $\therefore f(x+1)=\frac{9}{11}f(x)+\frac{150}{11}$. Solving we get, $f(x)=75+C(\frac{9}{11})^x$. For $x=0$, $f(x)=100$, $\therefore$ 25, and $f(x)=75+25(\frac{9}{11})^x$, [or as above $\frac{3}{4}+\frac{1}{4}(\frac{9}{11})^n$ dollars.]

Also solved by *J. H. DRUMMOND*, *J. K. ELLWOOD*, *F. P. MATZ*, and *G. B. M. ZERR*.

**22. Proposed by F. P. MATZ, M. A., M. Sc., Ph. D., Editor of the Department of Mathematics in the "New England Journal of Education" and Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

For the sum $D=\$30$, Messrs. Zerr and Ellwood contracted to plough the sod for a circular track, width, $m=60$ feet and inner radius $r=940$ feet. How is the money to be divided, if they commence ploughing at the inner circumference of the track, make uniform furrows of width $n=1\frac{1}{3}$ feet, and Mr. Ellwood continually follows Mr. Zerr during the ploughing?

**Solution by J. K. ELLWOOD, A. M., Pittsburg, Pennsylvania; M. A. GRUBER, A. M., Washington, D. C.; A. H. BELL, Hillsboro, Illinois; J. H. DRUMMOND, LL. D., Portland, Maine; and P S. BERG, Apple Creek, Ohio.**

$\frac{m}{n}=\frac{60}{1\frac{1}{3}}=45$ rounds; as Mr. Zerr begins, he plows 23 furrows and Mr. Ellwood 22 furrows.

Now, $\pi(r+n)^2-\pi n^2=\pi n(2r+n)$, the surface of $Z$'s 1st furrow.

His next furrow is represented by $\pi(r+3n)^2-\pi(r+2n)^2=\pi n(2r+5n)$, and so on, each furrow having just $4\pi n^2$ more surface than the next preceding. Hence, we have an arithmetical progression, whose first term is $\pi n(2r+n)$, com. diff. $4\pi n^2$, and number of terms 23. $\therefore$ The area of $Z$'s rounds is $\pi(r+n)^2-\pi r^2+\pi(r+3n)^2-\pi(r+2n)^2+\dots+\pi(r+45n)^2-\pi(r+44n)^2$, or $\pi n(2r+n)+\pi n(2r+5n)+\pi n(2r+9n)+\dots+\pi n(2r+89n)$, or $23\pi n(2r+45n)$.

Evidently, the area of $E$'s rounds is $\pi(r+2n)^2-\pi(r+n)^2+\pi(r+4n)^2-\pi(r+3n)^2+\dots+\pi(r+44n)^2-\pi(r+43n)^2$, or $\pi n(2r+3n)+\pi n(2r+7n)+\pi n(2r+11n)+\dots+\pi n(2r+87n)$, or $22\pi n(2r+45n)$.

Hence, $Z$'s portion of $D=\$30$ : $E$'s portion $=23\pi n(2r+45n):22\pi n(2r+45n),=23:22$.

$\therefore$ Zerr gets $\frac{2}{4}\frac{3}{5}$ $D=\frac{2}{4}\frac{3}{5}$ of \$30=\$15$\frac{1}{3}$, and Ellwood gets $\frac{2}{4}\frac{2}{5}$ $D=\frac{2}{4}\frac{2}{5}$ of \$30=\$14$\frac{2}{3}$.

Also solved by Professors *MATZ, PHILBRICK*, and *ZERR*.

NOTE: –H. W. Draughon remarks of Professor Zerr's solution of prob. 9, that the method fails unless the original equations are factored as in (1), (2), and (3) of solution; that they can not be so factored unless the values of $x$, $y$, and $z$ are known; and if these values are known, there is no need of solving. A similar comment has been received from Professor M. C. Stevens.

## PROBLEMS.

**29. Suggested by MANSFIELD MERRIMAN, C. E., Ph. D., Professor of Civil Engineering, Lehigh University, South Bethelehem, Pennsylvania.**

Solve neatly the equations: $\dfrac{y(1+x^2)}{x(1+y^2)}=a\ldots.(1)$, and $\dfrac{y^4(1+x^8)}{x^4(1+y^8)}=b..(2)$.

**30. Proposed by C. A. ROBERTS, Long Bottom, Ohio.**

Find the sum of $n=10$ terms of the series $1+15+55+134+265\ldots.$

**31. Proposed by D. G. DORRANCE, Jr., Camden, Oneida County, New York.**

Sum the series 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc. to $n$ terms. Also what is the $n$th term?

**32. Proposed by LEV. WEINER, Professor of Modern Languages, Missouri State University, Columbia, Missouri.**

Find a number consisting of 6 digits which when multipled by the first 6 natural numbers gives the same digits in rotation.

**33. Proposed by C. E. WHITE, Trafalgar, Indiana.**

Show that every algebraic equation of the $n$th degree, $n$ being greater than two, which is complete in its terms may be transposed into an infinite number of equations which want their second term.

Solutions to these problems should be received on or before November 1st.

# GEOMETRY.

**Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should besent to him.**

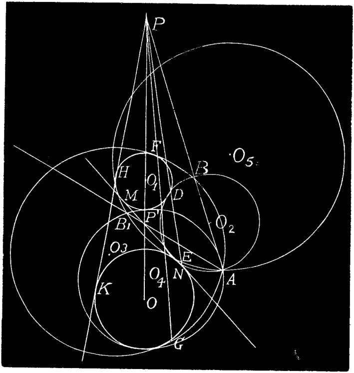
## SOLUTIONS TO PROBLEMS.

**14. Proposed by HENRY HEATON, M. S., Atlantic City, Iowa.**

Through a given point to draw four circles tangent to two given circles.

II. Solution by Professor G. B. M ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $A$ be the given point $O, O_1$, the given circle, draw $P\ O_1\ O$ the line joining the centres of the given circles and $PHK$ their common tangent. Let $AEDB$ be one of the circles satisfying the condition. Then $PA.PB=PO.PO_1=PK.PH=PE.PD$. Hence, join $A$ to the point $P$ and make $PA.PB=PK.PH$ this determines $B$; then a circle through $A, B$ tangent to $O_1$ or $O$




satisfies the conditions, for $PA.PB=PE.PD$; but two such circles $O_2, O_3$ can be drawn. [See problem 18.]

Similarly draw the internal common tangent $MN$. Let $P_1$ be the point of meeting of this tangent with $OO_1$, Take $P_1N.P_1M=P_1A.P_1B_1$. This determines $B_1$ and through $A, B_1$ two circles can be drawn satisfying the conditions. These are $O_4, O_5$.

**16.** Proposed by H. C. WHITAKER, B. S., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

Three lights of intensities 2, 4, and 5 are placed respectively at points the co-ordinates of which are (0,3) (4,5) and (9,0). Find a point in the plane of the lights equally illuminated by all of them.

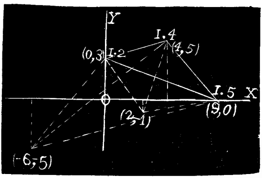
I. Solution by J. F. W. SCHEFFER. A. M., Hagerstown, Maryland, and the PROPOSER

$$\frac{2}{x^2+(y-3)^2}=\frac{4}{(x-4)^2+(y-5)^2}=\frac{5}{(x-9)^2+y^2}.$$

Taking the equation formed by the 1st and 2nd $x^2+8x+y^2-2y-23=0$, is the locus of points equally illuminated by the lights of 2 and 4 intensities. The 1st and 3rd give $x^2+12x+y^2-10y-39=0$ as being points equalluminated by lights 2 and 5. The 2nd and 3rd give $x^2+32x+y^2-50y-119=0$ for lights 4 and 5. The points equally illuminated by all are found by making these simultaneous as $(2,-1)$ and $(-6,-5)$.

Let $x$, $y$ be the co-ordinates of the point sought.

Then since intensity at units distance divided by the square of the distance to any point is the intensity at that point, we get




$$\frac{2}{(3-y)^2+x^2}=\frac{4}{(4-x)^2+(5-y)^2}$$

$$=\frac{5}{(9-x)^2+y^2}$$

$\therefore\ 39=x^2+y^2+12x-10y$ or

$x=-6\pm\sqrt{75-y^2+10y}$;

$23=x^2+y^2+8x-2y$ or

$x=-4\pm\sqrt{39-y^2+2y}$.

$\therefore\ -6\pm\sqrt{75-y^2+10y}=$

$-4\pm\sqrt{39-y^2+2y}$ or $y^2+6y=-5$.

$\therefore\ y=-1$ or $-5$, $x=2$ or $-6$.

$\therefore$ There are two points $(2,-1)$, $(-6,-5)$ both on the line $x-2y=4$.

Solutions to this problem were also received from *J. R. BALDWIN, P. H. PHILBRICK, LEONARD E. DICKSON*, and *T. W. ATKINSON*. An excellent solution accompanied by a beautiful blue-print figure was also received, but the author failed to sign his name to his work.

**17. Proposed by ROBERT J. ALEY, A. M., Professor of Mathematics in the Indiana University, Bloomington, Indiana.**

Draw a circle bisecting the circumference of three given circles.

**Solution by HENRY HEATON, M. S., Atlantic, Iowa.**

Let $A$ and $B$ be the centers of any two circles. Join $AB$ and draw diameters $DC$ and $EF$ perpendicular thereto. Through $P$, the center of the circle which would pass through $D$, $C$, $E$, and $F$, draw $QR$, perpendicular to $AB$.

From $L$, any point of $QR$, draw $LA$, $LB$, and the diameters $GH$ and $KI$ respectively perpendicular to $LA$ and $LB$. Then will $LC=LH=LI=LK$.

For $PD=PE$ by construction. $\therefore\ PA^2+AD^2=PB^2+BE^2$.

Substituting $AC^2$ for its equal $AD^2$, $BK^2$ for its equal $BE^2$, and adding $PL^2$ to both members of the equation, we get $PL^2+PA^2+AC^2=PL^2+PB^2+BK^2$. But the first member of this equation equals $LC^2$ and the second member equals $LK^2$.

$\therefore\ LC=LK$. But $LC=LH$ and $LK=LI$.

$\therefore\ LC=LH=LI=LK$. Since $GH$ and $IK$ are diameters, if a circle be passed through $G, H, I$, and $K$ it will bisect the circumferences of the circles $A$ and $B$.

Since $L$ is any point of $QR$, $QR$ is the locus of centers of circles which will bisect the circumferences of the circles $A$ and $B$.

The required construction is now obvious. Draw the locus of centers of circles which bisect the circumferences of the given circles, taken two and two. Their common intersection will be the center of the required circle.

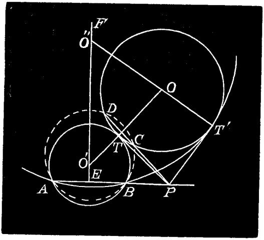
Solutions to this problem were also received from *Professors H. W. DRAUGHON, WILLIAM HOOVER, WILLIAM SYMMONDS, P. H., PHILBRICK* and ——.

**18. Proposed by Professor HENRY HEATON, Atlantic, Iowa.**

Through two given points to draw two circles tangent to a given circle.

**Solution by Professor T. A. TIMMONS, St. Mary's Kentucky; D. A. ROTHRICK. A M., Professor of Mathematics, Indiana University, Bloomington, Indiana; J. H. BEACH, Tiffin, Ohio; P. S. BERG, Apple Creek, Ohio; JOHN DOLLMAN, Jr., Councellor at Law, Philadelphia, Pennsylvania; and P. H. PHILBRICK, C. E., Lake Charles, Louisana**

Let $A$ and $B$ be the given points and $O$ the center of the given circle. Through the points $A$ and $B$ draw *any* circle cutting the given circle in the points $C$ and $D$. Draw the lines $AB$ and $DC$ and produce them until they meet in $P$. Draw the tangents $PT$ and $PT'$. At $E$, the middle point of $AB$, erect the perpendicular $EF$. Now $EF$ is the locus of the center of all circles passing through $A$ and $B$. Draw the radius $OT$ and produce it till it meets $EF$ in $O'$. Then $O'$ is the center of a circle passing through $A$ and $B$ and tangent to the given circle at $T$.




In like manner, by drawing the radius $OT'$ and producing it to meet $EF$, we find $O''$, the center of another circle fulfilling the conditions of the problem.

*Discussion.*—If one of the given points is within the given circle, the point $P$ falls within the given circle and there is no solution. If the given points lie on the circumference of the given circle there is one solution. In all other cases there are two solutions. When $EF$ passes, through the center of the given circle the general construction fails.

This problem was solved in a similar manner by *G. B. M. ZERR. WILLIAM SYMMONDS*, and *The PROPOSER. H. C. WHITAKER* solved it by Cartesian Geometry. *CHRISTIAN HOMING, Professor of Mathematics, in Herdleberg College*, did not solve the problem but referred to Casey's Sequel to Euclid's Elements. Prop. X., Sect. V., Bk. VI.

# CALCULUS.

Conducted by J. M. Colaw, Monterey, Va. All contributions to this department should be sent to him.

## SOLUTIONS TO PROBLEMS.

**18. Proposed by J. M. BANDY, M. A., Professor of Mathematics, Elon College, North Carolina.**

If the ordinate $ST$ of any point $T$ on a circle

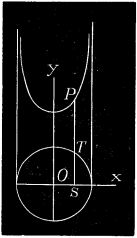
$$x^2+y^2=r^2$$

be produced so that $ST.TP=r^2$, prove that the whole area between the locus of $P$ and its asymptotes [and the circle] is double the area of the circle.

**I. Solution by M. C. STEVENS, M. A., Professor of Mathematics, Perdue University, Lafayette, Indiana; and COOPER D. SCHMITT, Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.**

First to find the locus of $P$: When $ST.TP=r^2$, $PS=y$, $OS=x$, $ST=\sqrt{r^2-x^2}$; $PT=y-\sqrt{r^2-x^2}$. $\therefore(y-\sqrt{r^2-x^2})(\sqrt{r^2-x^2})=r^2$; whence

$y=\dfrac{2r^2-x^2}{\sqrt{(r^2-x^2)}}$ locus of $P$. The asymptotes are $x=\pm r$.




$$A=\text{area}=4\int_0^r y\,dx=4\int_0^r\frac{2r^2-x^2}{\sqrt{(r^2-x^2)}}\,dx.$$

$$A=4\left\{\frac{x}{2}\sqrt{(r^2-x^2)}+\frac{3}{2}r^2\sin^{-1}\frac{x}{r}\right\}_0^r=3\pi r^2.$$

The area between the curve, its asymptotes and the circle (which was evidently meant)$=3\pi r^2-\pi r^2=2\pi r^2$.

**II. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics in the Ohio University, Athens, Ohio; and C. W. M. BLACK, A. M., Department of Mathematics, Wilmington Conference Academy, Dover, Delaware.**

Let the co-ordinates of $T$ be $(x',y')$ and those of $P$, $(x,y)$; then plainly $(y-y')y'=r^2\ldots.(1)$, and from the circle, $y'=\sqrt{(r^2-x^2}\ldots.(2)$.

Eliminating $y'$ from (1) and (2), we have the locus of $P$ given by

$$y=\sqrt{(r^2-x^2)}+\frac{r^2}{\sqrt{(r^2-x^2)}}=\frac{2r^2-x^2}{\sqrt{(r^2-x^2)}}.$$

$$A=\text{area}=4\int_0^r y\,dx=4\int_0^r\frac{2r^2-x^2}{\sqrt{(r^2-x^2)}}\,dx.$$

$$A=4\left\{\frac{x}{2}\sqrt{(r^2-x^2}+\frac{3}{2}r^2\sin^{-1}\frac{x}{r}\right\}_0^r=3\pi r^2.$$

The area between the curve, its asymptotes and the circle (which was evidently meant) $3\pi r^2-\pi r^2=2\pi r^2$.

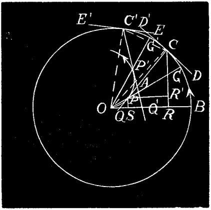
Also solved by *JOHN DOLMAN, Jr., ALFRED HUME, F. P. MATZ, J. SCHEFFER, H. C. WHITAKER*, and *G. B. M. ZERR*.

**19. Proposed by A. L. FOOTE, C. E., Merrick, Queens County, New York**

$A$ and $B$ are in a circular room $2R=30$ feet in diameter, $A$ being at the center and $B$ at the circumference. $B$ runs around at the rate of $v=600$ feet per minute and $A$ pursues him at the rate of $u=100$ feet per minute. How long will the race last, and how far will each have traveled till $B$ is caught?

**I. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virgini-.**

Let $O$ be the origin, $OB$ the axis of $x$, $B$ the starting point, $OB=a$, $OS=x$, $SP=y$, $OR=x_1$, $PC=m$, $OP=r$, $\angle COB=\psi$, $\angle POS=\theta$, $\angle CQS=\phi$.




$v$=velocity of pursued $u$= velocity of pursuer; $s$=arc $OP$; $P$, $P'$ two consecutive points on the pursuer's path; $\frac{v}{u}=n$. Then $ns=a\psi=a\cos^{-1}\frac{x_1}{a}=a\cos^{-1}\left(\frac{x+PR'}{a}\right)$.

$PR':CR'=QS:PS$. $\therefore$ $PR'=m\cos\phi=m\frac{dx}{ds}$. $\therefore$ $ns=a\cos^{-1}\left[\frac{x+m\frac{dx}{ds}}{a}\right]$,

or $a\cos\frac{ns}{a}=x+m\frac{dx}{ds}\ldots(1)$. But $a^2=r^2+m^2-2mr\cos OPC=r^2+m^2+2mr\cos(\phi-\theta)$. $\cos(\phi-\theta)=\frac{dr}{ds}$, $\therefore$ $a^2=r^2+m^2+2mr\frac{dr}{ds}$.

$$\therefore\ m=\sqrt{a^2-r^2+r^2\left(\frac{dr}{ds}\right)^2}-r\frac{dr}{ds},\ \text{but}\ \frac{ds}{dr}=\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2},$$

$$\frac{dy}{dx}=\frac{\sin\theta\frac{dr}{d\theta}+r\cos\theta}{\cos\theta\frac{dr}{d\theta}-r\sin\theta}.\qquad x=r\cos\theta,\ y=r\sin\theta,\ r=\sqrt{(x^2+y^2)}.$$

$$\therefore\ \frac{d\theta}{dr}=\frac{x\frac{dy}{dx}-y}{\sqrt{(x^2+y^2)}\left(x+y\frac{dy}{dx}\right)},\quad\text{and}\quad\frac{ds}{dr}=\frac{\sqrt{(x^2+y^2)\left[1+\left(\frac{dy}{dx}\right)^2\right]}}{x+y\frac{dy}{dx}}.$$

$$\therefore\ m=\frac{\sqrt{a^2\left[1+\left(\frac{dy}{dx}\right)^2\right]-\left(y-x\frac{dy}{dx}\right)^2}-\left(x+y\frac{dy}{dx}\right)}{1+\left(\frac{dy}{dx}\right)^2}.$$

$$\therefore\ a\cos\frac{ns}{a}=x+\frac{\sqrt{a^2\left[1+\left(\frac{dy}{dx}\right)^2\right]-\left(y-x\frac{dy}{dx}\right)^2}-\left(x+y\frac{dy}{dx}\right)}{1+\left(\frac{dy}{dx}\right)^2}.$$

If $CR=y_1$, $ns=a\sin^{-1}\frac{y_1}{a}=a\sin^{-1}\left(\frac{y+CR'}{a}\right)$,

but $CR'=m\sin\phi=m\frac{dy}{ds}$. $\therefore\ a\sin\frac{ns}{a}=y+m\frac{dy}{ds}\ldots.(2)$.

$$\therefore\ a\sin\frac{ns}{a}=y+\frac{\sqrt{a^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\}-\left(y+x\frac{dy}{dx}\right)^2}-\left(x+y\frac{dy}{dx}\right)}{\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}}.$$

The solution of either of these differential equations (if possible) give the rectangular equation to the path of the pursuer After we know the equato the curve we can find its length, from which we know the time and distance. If the pursuer's velocity is less than that of the pursued the race will last an infinitely long time, or the *pursued* will catch the *pursuer* and thus end the race.

II. **Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Let $O$ and $B$ represent the starting-points of $A$ and $B$, then will $P$ and $C$ represent their positions at any time after starting. Make $OB=R=15$ feet, $n=v\div u=6$, $\angle BOC=\psi$, $\angle BOG=\theta$, and $BQC=\phi$. Obviously the polar and the Cartesian co-ordinates of the point $P$ located on the required curve, are respectively $(r,\ \theta)$ and $(x,\ y)$. Hence $dy\div dx=\tan\phi$,

$OQ=y(dx\div dy)=y\cot\phi$, $\angle OPQ=(\phi-\theta)$, and $POC=(\psi-\theta)$.

Now $PQ=\left(\frac{\sin(\psi-\theta)}{\sin(\phi-\theta)}\right)OQ=y\operatorname{cosec}\phi\ldots.(1)$. That is, $\frac{\sin(\psi-\theta)}{\sin(\phi-\theta)}$

$=\tan\phi\operatorname{cosec}\phi\ldots.(2)$.

$\therefore\ \sin\psi$

$$=\frac{-(\tan\phi-\tan\theta)\pm\sqrt{\{(\tan\phi-\tan\theta)^2+(1+\tan^2\theta)[2\tan\phi\tan\theta-\tan^2\phi]\}}}{1+\tan^2\theta};$$

and $\psi=\sin^{-1}\left(\frac{(\tan\phi-\tan\theta)\pm\tan\theta\sqrt{(1+2\tan\phi\tan\theta-\tan^2\phi)}}{1+\tan^2\theta}\right)$,

$$=\sin^{-1}\left(\frac{(\tan\phi-\tan\theta)\pm\tan\theta\sqrt{[(1+\tan^2\theta)-(\tan\phi-\tan\theta)^2]}}{1+\tan^2\theta}\right).$$

But $s=\int\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$; $\tan\theta=y\div x$; and from the problem, $ns=R\psi\ldots.(3)$.

By obvious transformations, we have from (3) $\left(1+\frac{y^2}{x^2}\right)\sin\left(\frac{ns}{R}\right)=$

$\left\{\left(\frac{dy}{dx}-\frac{y}{x}\right)\pm\frac{y}{x}\sqrt{\left[\left(1+\frac{y^2}{x^2}\right)-\left(\frac{dy}{dx}-\frac{y}{x}\right)^2\right]}\right\}$, which is the Cartesian differen-

tial equation of the required curve; and this equation does not appear to be integrable. Many other differential equations of the required curve can be deduced; but all of these equations, as to their integrability, transcend the present limits of mathematical genius.

III. **Solution By JAMES McMAHON, M. A., Associate Editor of the "Annals of Mathematics", Department of Mathematics, Cornell University, Ithaca, New York.**

Let arc $BC=\nu$, $<DCQ=\mu$, $P$, $P'$ two consecutive points on the curve, then $dm=C'P'-CP=C'A-CA-(AP+AP')=CC'\cos\mu-\frac{u}{v}\ CC'$.

Let $\frac{u}{v}=n'$. $\therefore\ dm=d\nu(\cos\mu-n')$....(3). $d\mu=C'OC-CAC=\frac{d\nu}{a}-\frac{d\nu\sin\mu}{m}$.

$\therefore\ amd\mu=(m-a\sin\mu)d\nu$....(4). From (3) and (4) by elimination and solution (if possible) the equation to the curve of pursuit is found.

IV. **Comment by M. C. STEVENS, M. A., Professor of Mathematics, Purdue University, Lafayette, Indiana.**

To solve this problem, if we let $(x', y')$ be the co-ordinates of a point of the pursued at any moment, and $(x, y)$ the coordinates of the pnrsuer at the corresponding moment, then we have, $x'^2+y'^2=225$....(1). $y'-y=\frac{dy}{dx}(x'-x)$...(2)

$\sqrt{\left(\frac{dx'}{dx}\right)^2+\left(\frac{dy'}{dx}\right)^2}=6\sqrt{1+\left(\frac{dy}{dx}\right)^2}$....(3).

By elimination we can find a differential equation involving, $x$, $y$, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$. (See Boole's Diff. Equations, page 251).

But this results in a very complicated equation, which has never, sc far as I know, been solved. Now since the velocity of $B$ is greater than tha of $A$, $A$ will never overtake $B$; hence the answer to the question is, the *time* i *infinite* and the *distance* is *infinite*. (See remarks on Curve of Pursuit in Runkle's Mathematical Monthly, Vol. I, p. 248).

## PROBLEMS.

25. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Ne Windsor College, New Windsor, Maryland.**

The leaf of the curve: "The Devil on Two Sticks", equation $y^4-\text{ɔ}$ $+100a^2x^2-96a^2y^2=0$, revolves around the axis of $x$. Deduce the expression f the volume generated.

26. **Proposed by Professor J. F. W. SCHEFFER, M. A., Hagerstown, Maryland.**

According to Bessel the ratio of the squares of the polar diameter of t earth to that of the equatorial diameter is .9933254. Find at what altitude the a gle made by a body falling to the earth with a perpendicular to the surface is grea est. Find also this maximum angle.

Solutions to these problems should be received on or before November 1st.

# MECHANICS.

Conducted by B. F. FINKEL, Kidder, Mo. All Contributions to this department should be sent to him.

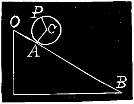
## SOLUTIONS TO PROBLEMS.

7. Proposed by DE VOLSON WOOD, M. A., M. Sc., C. E., Professor of Engineering, Stevens Institute of Technology, Hoboken, New Jersey.

A hollow sphere filled with frictionless water rolls down a rough plane whose length is $l$ and inclination $\theta$; when half way down the water suddenly freezes and adheres to the sphere. Required the time of the descent.

III. Solution by P. H. PHILBRICK, M. S., C. E., Lake Charles, Louisiana.

Let $m$ and $m'$ be the masses of the shell and of the water; $k$ and $k_1$ their radii of gyration about a diameter; and $a$ and $a_1$ the radii of the exterior and interior surfaces of the shell; $F$ the friction on the plane; $t_1$ the time of descent on the upper half of the plane, and $t_2$ the time of descent on the lower half; $V_1$ the velocity immediately before reaching the middle of the plane and $V_2$ the velocity immediately after passing the middle of the plane.




Take the axis of $x$ along the plane positive downward. For motion on the upper half of the plane we have, (see Wood's Analytical Mechanics) for translation, $(m+m')\dfrac{d^2x}{dt^2}=(m+m')g\sin\theta-F\ldots(1)$; and for rotation, $mk^2\dfrac{d^2\theta}{dt^2}=Fa\ldots(2)$, since the water does not rotate. Suppose the point $P$ on the shell to have been at the upper end of the plane upon starting; then $OA=\text{arc } AP$, or $x=a\theta$; then $dx=ad\theta$ and $d^2x=ad^2\theta$. Multiply (1) by $a^2$, (2) by $a$ and add, substituting $\dfrac{d^2x}{dt^2}$ for $a\dfrac{d^2\theta}{dt^2}$ and we have, $[(m+m')a^2+mk^2]\dfrac{d^2x}{dt^2}=(m+m')a^2g\sin\theta\ldots(4)$.

Integrating gives, $[(m+m')a^2+mk^2]\dfrac{dx}{dt}=(m+m')a^2g\sin\theta t\ldots(5)$.

$\therefore \dfrac{dx}{dt}=V=\dfrac{(m+m')a^2g\sin\theta t}{(m+m')a^2+mk^2}\ldots(6)$. Integrating (5) and putting $x=\frac{1}{2}l$ we easily find, $t_1=l^{\frac{1}{2}}\left[\dfrac{(m+m')a^2+mk^2}{(m+m')a^2g\sin\theta}\right]^{\frac{1}{2}}\ldots(7)$. From (6) and (7),

$$V_1=l^{\frac{1}{2}}\left[\frac{(m+m')a^2g\sin\theta}{(m+m')a^2+mk^2}\right]^{\frac{1}{2}}\ldots(8).$$

The energy of the system just before reaching the middle of the plane is, $\frac{1}{2}(m+m')V^2_1+\frac{1}{2}mk^2\left(\frac{V^2_1}{a}\right)$, and just after passing the middle of the plane is, $\frac{1}{2}(m+m')V^2_2+(\frac{1}{2}mk^2+\frac{1}{2}m'k^2_1)\left(\frac{V_2}{a}\right)^2$.

Equating these expressions which must be equal, we have,

$$V_2=V_1\left[\frac{(m+m')a^2+m'k^2}{(m+m')a^2+mk^2+m'k^2_1}\right]^{\frac{1}{2}}\ldots.(9).$$

Since the ice rotates with the shell, the equations of motion for the lower half of the plane are, $(m+m')\frac{d^2x}{dt^2}=(m+m')g\sin\theta-F\ldots(10)$, and $(mk^2+m'k^2_1)\frac{d^2\theta}{dt^2}=Fa\ldots.(11)$. Then as before,

$$[(m+m')a^2+mk^2+m'k^2_1]\frac{d^2x}{dt^2}=(m+m')a^2g\sin\theta\ldots.(12).$$

Integrating gives, $[(m+m')a^2+mk^2+(m+m')k^2_1]\frac{dx}{dt}$

$$=(m+m')a^2g\sin\theta t+C\ldots.(13).$$

But $\frac{dx}{dt}=V_2$ for $t=0$. $\therefore$ $C=[(m+m')a^2+mk^2+m'k^2_1]V_2$ and (13) becomes, $[(m+m')a^2+mk^2+m'k^2_1]=(m+m')a^2g\sin\theta t+[(m+m')a^2+mk^2+m'k^2_1]V_2\ldots.(14)$. Integrating, putting $x=\frac{1}{2}l$ and $t=t_2$ gives, $[(m+m')a^2+mk^2+m'k^2_1]l=(m+m')a^2g\sin\theta t^2_2+2[(m+m')a^2+mk^2+m'k^2_1]V_2t_2$.

For brevity, put the coeffecients of $t_2$ and $t^2_2$ equal to $c$ and $d$ respectively, and the absolute term equal to $b$. Then $b=dt^2_2+ct_2$ or $t_2=\frac{1}{2b}(-c+\sqrt{c^2+4bd}\ldots.(15)$. Equations (7) and (15) give $T=t_1+t_2=$the total time.

**IV. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

Let $x=$the distance the shell and water of masses $m$, $m'$ respectively have moved down the plane in the time $t$ from the beginning of motion, $F=$ the friction, $r$, $r'$ the exterior and interior radii of the shell, $k$, $k'$ the radii of gyration of $m$ and $m'$, and $\phi=$ the amount of rotation of the shell.

Resolving parallel to the plane, and taking moments about the center of the shell,

$$(m+m')\frac{d^2x}{dt^2}=(m+m')g\sin\theta-F\ldots.(1),\text{and } mk^2\frac{d^2\theta}{dt^2}=Fr\ldots.(2).$$

We have also from the geometry, $x=r\phi\ldots.(3)$. From (3), $\frac{d^2\theta}{dt^2}=\frac{1}{r}\frac{d^2x}{dt^2}..(4)$.

This in (2) gives $mk^2 \frac{d^2x}{dt^2}=Fr^2 \ldots(5)$. Eliminating $F$ from (1) and (5), $[(m+m')r^2+mk^2]\frac{d^2x}{dt^2}=(m+m')r^2 g\sin\theta \ldots(6)$.

Integrating (6) twice, noticing that initially $\frac{dx}{dt}=0$, $x=0$, we have $[(m+m')r^2+mk^2]\frac{dx^2}{dt^2}=2(m+m')r^2 g\sin\theta . x \ldots(7)$, and $[(m+m')r^2+mk^2)]x =\frac{1}{2}(m+m')r^2 g\sin\theta . t^2 \ldots(8)$. When $x=\frac{l}{2}$, these give $v=\frac{dx}{dt}=$

$$\sqrt{\frac{(m+m')r^2 g\sin\theta . l}{(m+m')r^2+mk^2}}\ldots(9) \text{ and } t_1=\sqrt{\frac{[(m+m')r^2+mk^2]l}{(m+m')r^2 g\sin\theta}}\ldots(10).$$

The circumstances of motion changing at this point, it is necessary to determine the instantaneous change in $v$ and $\omega$, the latter being the value of $\frac{d\phi}{dt}=\frac{1}{r}\frac{dx}{dt}\ldots(11)$ from (3).

Assuming the principle of the conservation of the moment of momentum as holding here,

$mk^2\omega+(m+m')vr=mk^2\omega'+m'k'^2\omega'+(m+m')v'r\ldots(11)$, $v$ having changed to $v'$, and $\omega$ to $\omega'$. But $v=r\omega$, $v'=r'\omega'$, $k^2=\frac{2}{5}\frac{r^5-r'^5}{r^3-r'^3}$, $k'^2=\frac{2}{5}r'^2\ldots(12)$.

These equations give

$$v'=\frac{2m(r^5-r'^5)+5(m+m')(r^3-r'^3)r^2}{2m(r^5-r'^5)+2m'r'^2(r^3-r'^3)+5(m+m')(r^3-r'^3)}\,v\ldots(13).$$

If $y$= the distance passed over from the middle of the plane after any time $t$, and $\phi'$ the corresponding amount of rotation, we have, resolving as before,

$$(m+m')\frac{d^2y}{dt^2}=(m+m')g\sin\theta-F\ldots(14),\quad (mk^2+m'k'^2)\frac{d^2\phi'}{dt^2}=Fr\ldots(15).$$

We have also $y=r\phi'\ldots(16)$.

Eliminating $F$ from (14) and (15) and using (16), there is $[(m+m')r^2 + mk^2+m'k'^2]\frac{d^2y}{dt^2}=(m+m')r^2 g\sin\theta\ldots(17)$.

Integrating (17) twice, and noticing that initially $\frac{dy}{dt}=v'$, $y=0$, we have

$$[(m+m')r^2+mk^2+m'k'^2]y=\tfrac{1}{2}(m+m')r^2 g\sin\theta . t^2+[(m+m')r^2 +mk^2+m'k'^2]v'\ldots(18).$$

Putting $y=\frac{1}{2}l$, we find the time $t_2$ for the lower half of the plane, and then the required time $=t_1+t_2$.

# DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

## SOLUTIONS TO PROBLEMS.

**6. Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

Find three whole numbers the sum of any two of which is a cube.

**II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**

Let $x$, $y$, and $z$=the numbers. Then $x+y=a^3$, $x+z=b^3$. and $y+z=c^3$; in which $c>b>a$. Hence $x+y+z=\dfrac{a^3+b^3+c^3}{2}$.

$$\therefore\ x=\frac{a^3+b^3+c^3}{2}-c^3;\ y=\frac{a^3+b^3+c^3}{2}-b^3;\ z=\frac{a^3+b^3+c^3}{2}-a^3.$$

In order that each of the numbers be integral, *either* each of the cubes must be even, *or* two of the cubes must be odd and one even. Also, in order that each of the numbers be positive, $a^3+b^3>c^3$.

The simple rule for finding the three numbers is as follows: Take three cubes fulfilling the above two conditions, and from half their sum subtract each cube separately; the remainders will be the three numbers required.

It is evident that many sets of numbers be obtained. We will illustrate by the following consecutive cubes: 1, (2) 8, (3) 27, (4) 64, (5) 125, (6) 216, (7) 343, (8) 512, (9) 729, (10) 1000, (11) 1331, (12) 1728, (13) 2197, (14) 2744, (15) 3375, (16) 4096, (17) 4913, (18) 5832.

The first three cubes answering the above conditions are 343, 512, and 729. $\frac{1}{2}$ their sum is 792; $x=792-729=63$; $y=792-512=286$; $z=792-343=449$. The next three cubes are $(7)^3$, $(9)^3$, and $(10)^3$. The first three *even* cubes are $(12)^3$, $(14)^3$, and $(19)^3$.

**9. Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.**

It is required to find three numbers the sum of whose 4th powers is a square.

**I. Solution by R. J. ADCOCK, Larchland, Warren County, Illinois.**

If $u=x^2+y^2-z^2$, then $u^2=x^4+y^4+z^4+2x^2y^2-2x^2z^2-2y^2z^2$, in which if $2x^2y^2-2x^2z^2-2y^2z^2=0$, $z^2=\dfrac{x^2y^2}{x^2+y^2}$, and $u=x^2+y^2-\dfrac{x^2y^2}{x^2+y^2}$

$$=\frac{x^4+y^4+x^2y^2}{x^2+y^2};\ u^2=x^4+y^4+z^4=x^4+y^4+\frac{x^4y^4}{(x^2+y^2)^2}=\left(\frac{x^4+y^4+x^2y^2}{x^2+y^2}\right)^2.$$

$\therefore\ x^4(x^2+y^2)^2+y^4(x^2+y^2)^2+x^4y^4=(x^4+y^4+x^2y^2)^2$, which is a general equation for the sum of the 4th powers of three quantities=a square, when $x^2+y^2$=a square. Making $x=3, y=4$, $x^2+y^2=5^2$, and $3^4\times5^4+4^4\times5^4+3^4\times4^4=15^4+20^4+12^4=(3^4+4^4+12^4)^2=481^2=231361$.

**II. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.**

Let $ax$, $ay$, and $az$ denote the numbers required. Then must $a^4x^4+a^4y^4+a^4z^4=\square$, or, rejecting the square factor $a^4$, we must make $x^4+y^4+z^4=\square=(x^2-y^2+z^2)^2$ say; whence $x^2z^2=x^2y^2+y^2z^2=y^2(x^2+z^2)$. Assume

$x=p^2-q^2$ and $z=2pq$, and we have $xz=y(p^2+q^2)$, which gives $y=\dfrac{xz}{p^2+q^2}$

$=\dfrac{2pq(p^2-q^2)}{p^2+q^2}$. Now take $a=p^2+q^2$ and we have $ax=p^4-q^4$, $ay=2pq(p^2-q^2)$, $az=2pq(p^2+q^2)$.

$$\therefore\ (p^4-q^4)^4+[2pq(p^2-q^2)]^4+[2pq(p^2+q^2)]^4=[(p^4-q^4)^2+12p^4q^4]^2=[(p^2+q^2)^4-4p^2q^2(p^2-q^2)^2]^2.$$

Take $p=2$, $q=1$ and we find the numbers 12, 15 and 20, the sum of whose 4th powers$=(481)^2$. These are the *smallest* numbers answering the problem, but an infinite number of other answers may be found by varying the values of $p$ and $q$.

Also solved by *H. W. DRAUGHON, A. L. FOOTE, F. P. MATZ*, and *G. B. M. ZEER*.

**10. Proposed by L. B. HAYWARD, Bingham, Ohio.**

Find two numbers, such that each of them and also their sum and their difference when increased by unity shall all be square numbers.

**I. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.**

The leaving out of commas leaves one in doubt whether each of the numbers *plus one*, or simply each of the numbers, must be a square. The more probable construction requires $x+1$ (1), $y+1$ (2), and $x\pm y+1$ (3) to be squares. Take $x+1=m^2+n^2$ and $y=2mn$ and the two conditions of (3) are met. But $m^2+n^2$ must be a square: this happens when $m=p^2-q^2$ and $n=2pq$. It now remains to make $2mn+1$, or $4pq(p^2-q^2)+1=\square=$(say) $(2pq\pm1)^2$. Reducing we have $p^2-qp=(q^2\pm1)$, from which $p=\dfrac{q\pm\sqrt{5q^2\pm4}}{2}$. $5q^2+4=\square=(sq\pm2)^2$;

then $q=\pm\dfrac{4s}{s^2-5}$. We readily find $p=\dfrac{\pm2s\pm(s^2+5)}{s^2-5}$.

$s=1$, $q=\pm1$; $p=\pm1$, or $\pm2$, $q=\pm1$, $p=\pm2$, $m=3$, and $n=4$ and $y=24$, and $x=24$. $s=2$, $q=\pm8$, $p=\pm5$ or $\pm13$; $m=39$ and $n=80$; or $m=105$ and $n=208$; $x=7920$ or 54288; $y=6240$ or 43680. $s=3$, $q=\pm3$, $p=\pm5$ or $\pm2$; $x=1155$ or 168; $y=960$ or 120; these last values of $x$ and $y$ are the least integrals which I have obtained. The reduction of $\sqrt{(5q^2-4)}=\square$ is quite interesting, but as it gives no values for $x$ and $y$, different from those already obtained, I omit it.

**II. Solution by A. L. FOOTE, C. E., Merrick, New York, and P. S. BERG, Apple Creek, Ohio.**

Solving under the other construction, let $x^2$ and $2x$ be the required numbers; then the sum increased by unity is $x^2+2x+1$ which is a square; and their difference increased by unity is $x^2-2x+1$ which is also a square,

and since $x^2$ is a square it only remains to make $2x$ a square, which it is when $x=2$. But this value of $x$ makes the numbers the same.

Our next value of $x$ is 8 and $x^2=64$, and $2x=16$, which numbers answer the conditions. The next value of $x$ is 18 and the numbers are 324 and 36, and so on *ad infinitum*.

Also solved by *W. H. DRAUGHON, ARTEMAS MARTIN, F. P. MATZ, J. F. W. SCHEFFER, G. B. M. ZERR*, and *J. K. ELLWOOD*.

## PROBLEMS.

13. **Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.**

It is is required to find four numbers the sum of whose fourth powers is a square number.

14. **Proposed by SYLVESTER ROBINS, North Branch Depot, New Jersey.**

Find initial terms in each of three infinite series of prime, integral, rational, scalene triangles, where 9 shall be the base, and the other two sides of every term shall have a constant difference.

15. **Problems, or Propositions by M. A. GRUBER, M. A, War Department, Washington, D. C.**

($a$) The *difference* of two *odd* squares is always divisible by 8. Corollary: Every odd square is of the form $8a+1$.

($b$) The *sum* of two *odd* squares is two times an *odd* number.

Solutions to these problems should be received on or before November 1st.

# AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Kidder, Mo. All Contributions to this department should be sent to him.

## SOLUTIONS TO PROBLEMS.

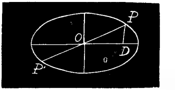
6. **Proposed by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.**

Find the average length of all the diameters that can be drawn in a given ellipse.

II. **Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, and the PROPOSER.**

By using the *complement* of the eccentric angle we deduce $OD=a\sin\phi$

$PD=b\cos\phi$, $PP'=2a\surd(1-e^2\cos^2\phi)$=any diameter, and $ds=\surd(1-e^2\sin^2\phi)d\phi$. Hence the required average length becomes




$$D=2a^2\int_0^{\frac{1}{2}\pi}\surd[(1-e^2\cos^2\phi)(1-e^2\sin^2\phi)]d\phi$$

$$\div a\int_0^{\frac{1}{2}\pi}\surd(1-e^2\sin^2\phi)d\phi$$

$$=\frac{2a}{E(e,\frac{1}{2}\pi)}\int_0^{\frac{1}{2}\pi}\surd[1-e^2(\sin^2\phi+\cos^2\phi)+e^4\sin^2\phi\cos^2\phi]d\phi$$

$$=\frac{a(2-e^2)}{E(e,\frac{1}{2}\pi)}\int_0^{\frac{1}{2}\pi}\sqrt{\left[1-\left(\frac{e^2}{2-e^2}\right)^2\cos^2 2\phi\right]}d\phi$$

$$=\frac{-\frac{1}{2}a(2-e^2)}{E(e,\frac{1}{2}\pi)}\int_{+\frac{1}{2}\pi}^{-\frac{1}{2}\pi}\sqrt{\left[1-\left(\frac{e^2}{2-e^2}\right)^2\sin^2(\tfrac{1}{2}\pi-2\phi)\right]}d(\tfrac{1}{2}\pi-2\phi)$$

$$=\frac{\frac{1}{2}a(2-e^2)}{E(e,\frac{1}{2}\pi)}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\sqrt{\left[1-\left(\frac{e^2}{2-e^2}\right)^2\sin^2\theta\right]}d\theta$$

$$=\frac{a(2-e^2)}{E(e,\frac{1}{2}\pi)}\int_0^{\frac{1}{2}\pi}\sqrt{\left[1-\left(\frac{e^2}{2-e^2}\right)^2\sin^2\theta\right]}d\theta\ldots.(1).$$

Representing by $c$ the modulus of the elliptic integral in (1), we have $D=a(2-e^2)[E(c,\frac{1}{2}\pi)\div E(e,\frac{1}{2}\pi)]$, which is the average length required.

**III. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

Let $2r$=any diameter.

Then $2r=2\sqrt{x^2+y^2}=2\surd\overline{b^2+e^2x^2}$ since $y^2=\dfrac{b^2}{a^2}(a^2-x^2)$ and $\dfrac{a^2-b^2}{a^2}=e^2$.

$$\therefore\ \triangle=\text{average length}=2\int_0^a\surd\overline{b^2+e^2x^2}dx\div\int_0^a dx$$

$$\triangle=\frac{2}{a}\int_0^a\sqrt{b^2+e^2x^2}dx=\frac{2}{a}\left[\frac{x}{2}\surd\overline{b^2+e^2x^2}+\frac{b^2}{2e}\log\left\{x+\frac{\surd\overline{b^2+e^2x^2}}{e}\right\}\right]_0^a$$

$$\therefore\ \triangle=a+\frac{b^2}{ae}\log\left\{\frac{a(1+e)}{b}\right\}.$$

NOTE—Professor Matz gave a similar solution obtaining the same result in a different form. He has now furnished six different solutions to this problem. Three of these solutions give, for $e=\frac{1}{2}$, the average length of a diameter $=\dfrac{263}{250}b$, two give the average$=\dfrac{228}{250}a$, the result in the above solution, and one gives average$=\dfrac{268}{250}b$. These results are due to different interpretations of the problem. It occurs to us that the correct solution is obtained by considering the number of diameters proportional to the circumference of the ellipse. Taking

any diameter and having it pass through all possible values within proper limits by varying the ordinate or abscissa will give the sum of all the diameters. Dividing the sum of all the diameters by the number, which is equal to the circumference of the ellipse, will give the average diameter. The difference in the results $\frac{263}{250}b$ and $\frac{228}{250}a$ is about $\frac{1}{1000}b$. ED.]

**7. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

A letter is known to have come either from *Oshkosh* or *Ashland*. The only two consecutive letters legible on the postmark are *SH*. What is the probability that the letter received came from Oshkosh?

**I. Solution by the PROPOSER.**

Of the six pairs of consecutive letters in the word Oshkosh, *SH* are 2 pairs. Hence if the letter came from Oshkosh, the probability that *SH* was the legible pair is $\frac{2}{6}$, or $\frac{1}{3}$. If the letter came from Ashland, this probability is $\frac{1}{6}$. The *a posteriori* probability that the letter received came from Oshkosh, is, therefore, $P_O=\frac{\frac{2}{6}}{\frac{2}{6}+\frac{1}{6}}=\frac{2}{3}$; and that it came from Ashland, is $P_A=\frac{\frac{1}{6}}{\frac{2}{6}+\frac{1}{6}}=\frac{1}{3}$.

$\therefore\ P_O+P_A=\frac{2}{3}+\frac{1}{3}=1$.

*Note.*—In this connection, the following problem is appropriate and interesting: A letter is known to have come either from *Sing Sing* or *Lansing*. The only four consecutive letters legible on the postmark are *SING*. What is the probability that the letter received came from Sing Sing?

*Answer*: $P_S+P_L=\frac{8}{13}+\frac{5}{13}=1$.

**II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**

According to the arrangement of the postmarks, *sh* of Ashland and the first *sh* of Oshkosh would be found in the left portion of the postmark, and the last *sh* of Oshkosh would be found in the right portion or near the top of the postmark.

There will, therefore, be two cases:—

(1) When a letter or figure of the date indicates the position of postmark.

(2) When the position of postmark cannot be determined.

*Case 1.* If *sh* is found in the right portion of postmark, the chances in favor of *Oshkosh* are $\frac{1}{0}$, or *infinity*; *i. e.* the letter came from Oshkosh.

If *sh* is found in the left portion, since the names of both places have the same number of letters and *sh* in both names is preceded only by the initial letter, the chances are equally divided, or $\frac{1}{1}$.

*Case 2.* In this case, since there are two *sh*'s in Oshkosh and only one in Ashland, the chances in favor of Oshkosh are 2 to 1.

**III. Solution by G. B. M. ZERR, Principal of High School, Staunton, Virginia.**

Since *sh* is found twice in Oshkosh and once in Ashland, the probability that the word is Oshkosh is $\frac{2}{3}$.

## PROBLEMS.

**15. Proposed by A. L. FOOTE. No. 80, Broad St., New York.**

A person 30 years of age has an annuity for 10 years, the present worth of which is $1000, provided he lives but ten years; for, if he dies, the annuity ceases. What is the annuity worth, on the assumption that 75 out of every 4385 persons die annually, between the ages 30 and 40 years?

**16. Proposed by H. W. DRAUGHON, Clinton, Louisiana.**

The probability that $A$ will speak the truth is twice the probability that $B$ will, in an independent statement, speak the truth; but, if $A$ exerts his influence, the probability is that $B$ will agree with him in any statement. What is the probability of the truth of their concurrent testimony, the chances being equal that $A$ may or may not be interested in the matter?

**17. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Find the average area of the circle which is the locus of the middle points of all chords passing through a point taken at random in the surface of a given circle.

Solutions to these problems should be received on or before November 1st.

# MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

## SOLUTIONS TO PROBLEMS.

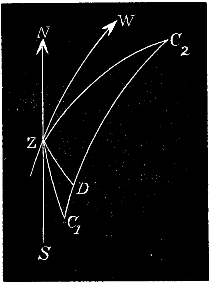
**10. Probosed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, Yates County, N. Y.**

A small cloud in the S. E. and altitude 70°, was soon after N. 60°E with an altitude of 30°. In what direction was the wind blowing, the track of the cloud being the arc of a great circle?

**I. Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Let $Z$ represent the zenith of the observer, $C_1$ and $C_2$ the positions of the cloud, and $ZD$ the perpendicular $C_1C_2$. The $\angle NZW$ represents the direction of the wind with respect to the observer and his zenith. Since $C_1Z=20°$, $C_2Z=60°$, $\angle C_1ZC_2=75°$, $\angle C_1ZS=45°$, and $\angle C_2ZN=60°$, the spherical triangle $C_1ZC_2$ gives $\angle C_1C_2Z=\phi=\tan^{-1}\left(\frac{\cos 20°}{\cos 40°}\cot 37\tfrac{1}{2}°\right)$

$-\tan^{-1}\left(\frac{\sin 20^\circ}{\sin 40^\circ}\cot 37\tfrac{1}{2}^\circ\right)=23^\circ\ 14'\ 3''.7083$; and the spherical triangle $C_2ZD$ gives $\angle C_2ZD=\psi=\cot^{-1}(\cos 60^\circ\div\cot\phi)$ $=77^\circ\ 53'\ 6''$. Obviously *the direction of the wind*, represented by $\angle NZW$, $=C_2ZD+\angle C_2ZN-90^\circ=\psi+60^\circ-90^\circ=\psi-30^\circ$, $=$N. $47^\circ 53' 6''$E.; for at $D$ the zone of the air in motion moves at right angles to $ZD$, and from $D$ to $Z$ the same zone moves also parallel with itself.




**II. Solution by G. B. M. ZERR, Principal of High School, Staunton, Virginia.**

Let $A$ be the station of the observer on the earth, $Z$ his zenith, $MZN$ his meridian. $B$, $C$ the positions of the cloud at first and last observation, $KBCP$ the cloud's track, $FG$ a plane through $A$ perpendicular to $ZA$, $D$, $E$ the projections of $B$, $C$ on this plane.

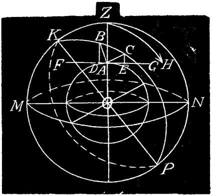
Let $AD=m$, $AE=n$, $AZ$, $AG$ the $X$, $Y$ axes. The plane of the cloud with reference to the observer passed through the points $A$, $B$, $C$.

$\therefore$ coordinates of $A$, $(0, 0, 0)$;

coordinates of $B$, $(m\tan 70^\circ, -m\cos 45^\circ, m\sin 45^\circ)=\left(m\tan 70^\circ, -\frac{m}{\sqrt{2}}, \frac{m}{\sqrt{2}}\right)$;

coordinates of $C$, $(n\tan 30^\circ, n\cos 60^\circ, n\sin 60^\circ)=\left(\frac{n}{\sqrt{3}}, \frac{n}{2}, \frac{n\sqrt{3}}{2}\right)$.

$$\therefore \begin{vmatrix} x, & y, & z, & 1 \\ 0, & 0, & 0, & 1 \\ m\tan 70^\circ, & -\frac{m}{\sqrt{2}}, & \frac{m}{\sqrt{2}}, & 1 \\ \frac{n}{\sqrt{3}}, & \frac{n}{2}, & \frac{n\sqrt{3}}{2}, & 1 \end{vmatrix}=0,$$ is the equation of this plane.




The direction of the wind is represented by $ZH$ parallel to $KBCP$ through the zenith $Z$.

Let $x=0$, and we get

$$\begin{vmatrix} 0, & y, & z, & 1 \\ 0, & 0, & 0, & 1 \\ m\tan 70^\circ, & -\frac{m}{\sqrt{2}}, & \frac{m}{\sqrt{2}}, & 1 \\ \frac{n}{\sqrt{3}}, & \frac{n}{2}, & \frac{n\sqrt{3}}{2}, & 1 \end{vmatrix}=0,$$ is the equation of the trace of the plane $ZHO$ on the plane $YZ$.

Writing out the determinant we get $(3\sqrt{2}\tan 70^\circ - 2)y = (\sqrt{6}\tan 70^\circ + 2)z$.

$\therefore\ \theta = \angle HZN = \tan^{-1}\left\{\dfrac{3\sqrt{2}\tan 70^\circ - 2}{\sqrt{6}\tan 70^\circ + 2}\right\} = 47^\circ\ 53'\ 6.''6$.

$\therefore$ N. $47^\circ\ 53'\ 6.''$ E. is the direction of the wind.

Also solved by *A. H. BELL*, and the *PROPOSER*.

## PROBLEMS.

**16. Yale Senior Prize Problem.---Contributed by H. A. NEWTON, LL. D., Professor of Mathematics, Yale College, New Haven, Connecticut.**

The axes of two right cylinders whose bases are circles of 4 and 6 inches radius respectively, intersect at right angles; compute to four decimal places the lengths of the curves of intersection of the two surfaces.

**Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, Yates county, New York.**

A bright star passed my meridian at 7 P. M. The Chronometer soon after ran down and stopped, but I set it again when the same star had a true altitude of $30^\circ = \alpha$. What time was it then, my latitude being $42^\circ 30'$N. $= \lambda$, and the star's Declination $60^\circ$N. $= \delta$ ?

Solutions to these problems should be received on or before November 1st.

# QUERIES AND INFORMATION.

**Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.**

**6. A reply to Professor WHITAKER'S Comment, by H. W. DRAUGHON.**

Professor Whitaker's explanation of the difficulty in L. B's solution seems to me illogical. L. B's answer is correct, and does prove if properly substituted.

Professor Whitaker does not discriminate between the sign indicating operation, placed before an expression, and the sign of the *value* of that expression found by solution. For instance, let us take the equation under discussion, $x - 4 = +\sqrt{x-4} + 4 \ldots . (1)$.

Squaring, we readily find, $\sqrt{x-4} = -1$. The $+$ sign before $\sqrt{x-4}$ in (1), merely indicates that the value of $\sqrt{x-4}$, be it positive or negative, is to be added to 4. If Professor Whitaker insists that $\sqrt{x-4}$ cannot have a negative value, he must deny that $x$ can have a negative value in the following equation: $x^2 + 2x = 3 \ldots . (2)$. To illustrate, let the value of $x^2$ be required; we readily find $x^2 = 9$ and $x^2 = 1$.

From the first result, we obtain $x = 3$, which does not satisfy (2), but